XVI.

On Certain Forms of Interpolation.

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THE quantity to be interpolated being a function of t, the following scheme shows the notation and arrangement adopted for a series of its values and their *differences.*

$$(1)\quad \begin{cases} f(t_0-1) = Y_{-1} & & \Delta^2_{-1} & & & \\ & \Delta_{-\frac{1}{2}} & & \Delta^3_{-\frac{1}{2}} & & \\ f(t_0) \quad\;\; = Y_0 & & \Delta^2_0 & & \Delta^4_0 & \\ & \Delta_{\frac{1}{2}} & & \Delta^3_{\frac{1}{2}} & & \Delta^5_{\frac{1}{2}} \\ f(t_0+1) = Y_1 & & \Delta^2_1 & & \Delta^4_1 & \\ & \Delta_{\frac{3}{2}} & & & & \\ f(t_0+2) = Y_2 & & & \text{\&c.,} & & \end{cases}$$

in which

$$(2)\quad \begin{cases} \Delta_{n+\frac{1}{2}} = Y_{n+1} - Y_n, & \Delta^3_{n+\frac{1}{2}} = \Delta^2_{n+1} - \Delta^2_n, \\ \Delta^2_n \quad\; = \Delta_{n+\frac{1}{2}} - \Delta_{n-\frac{1}{2}}, & \Delta^4_n \quad\; = \Delta^3_{n+\frac{1}{2}} - \Delta^3_{n-\frac{1}{2}}, \quad \text{\&c.} \end{cases}$$

Whenever the subscript numbers are omitted in writing differences, $+\frac{1}{2}$ is to be understood for all odd orders, and 0 for all even ones.

It may be observed that, if we put $\Delta_{-\frac{1}{2}}$ in place of Y_0, we shall have the following relations between *its* differences and the original ones of Y_0 :

$$(3)\quad \begin{cases} \Delta_{\frac{1}{2}}(\Delta_{-\frac{1}{2}}) = \Delta^2_0, \qquad\qquad \Delta^2_0(\Delta_{-\frac{1}{2}}) = \Delta^3_{-\frac{1}{2}}, \quad \text{\&c.;} \\ \text{and similarly, if we put } \Delta^2_0 \text{ in place of } Y_0, \\ \Delta_{\frac{1}{2}}(\Delta^2_0) \;\; = \Delta^3_{\frac{1}{2}}, \qquad\qquad \Delta^2_0(\Delta^2_0) = \Delta^4_0, \quad \text{\&c.;} \end{cases}$$

and so on with higher orders of differences, the law of this *symbolic multiplication* being to add the subscript numbers of the *factors* to form that of the *product,* as well as their (exponential) indices to form its index.

Let i be a positive integer, and $k=\frac{1}{i}$; then after the insertion between Y_n and Y_{n+1} of $i-1$ new values of the function (which is called *interpolating to* k^{ths}), let a portion of the series and its differences be

$$(4)\quad \left\{\begin{array}{lllll}
y_{-1} & & & & \\
 & \vartheta_{-\frac{1}{2}} & & & \\
Y_0 & & \delta_0^2 & & \delta_0^4 \\
 & \vartheta_{\frac{1}{2}} & & \vartheta_{\frac{1}{2}}^3 & \cdot \\
y_1 & & \vartheta_1^2 & & \cdot \\
 & \vartheta_{\frac{3}{2}} & & \vartheta_{\frac{3}{2}}^3 & \cdot \\
y_2 & & \vartheta_2^2 & & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
y_{i-1} & & \vartheta_{i-1}^2 & & \\
 & \vartheta_{i-\frac{1}{2}} & & \vartheta_{i-\frac{1}{2}}^3 & \\
Y_1 & & \delta_1^2 & & \\
 & \vartheta_{i+\frac{1}{2}} & & \&c. & \\
y_{i+1} & & & &
\end{array}\right.$$

The quantity ϑ will be an implicit function of t, the values indicated in (4) corresponding to $t=t_0-\frac{1}{2}k$, $t_0+\frac{1}{2}k$, $t_0+\frac{3}{2}k$, &c.; let its value for $t=t_0\pm\frac{1}{2}$ be $\delta_{\pm\frac{1}{2}}$. δ_0^2 will then, *in all cases, whether i is odd or even,* be the same function of $\delta_{-\frac{1}{2}}$ and *its* differences that $\delta_{\frac{1}{2}}$ is of Y_0 and its differences; that is, if

$$(5)\quad \left\{\begin{array}{l}
\text{then} \qquad \begin{aligned}\delta_{\pm\frac{1}{2}} &= \varphi_{\pm\frac{1}{2}}(Y_0),\\ \delta_0^2 &= \varphi_{\frac{1}{2}}(\delta_{-\frac{1}{2}}) = \varphi_{\frac{1}{2}}\cdot\varphi_{-\frac{1}{2}}(Y_0) = \varphi_0^2(Y_0);\end{aligned}\\
\text{and similarly for similar functions of higher orders:}\\
\qquad \begin{aligned}\delta^3_{\pm\frac{1}{2}} &= \varphi_{\pm\frac{1}{2}}(\delta_0^2) = \varphi_{\pm\frac{1}{2}}\cdot\varphi_{\frac{1}{2}}\cdot\varphi_{-\frac{1}{2}}(Y_0) = \varphi^3_{\pm\frac{1}{2}}(Y_0),\\ \delta_0^4 &= \varphi_{\frac{1}{2}}(\delta^3_{-\frac{1}{2}}) = \varphi_{\frac{1}{2}}\cdot\varphi_{-\frac{1}{2}}\cdot\varphi_{\frac{1}{2}}\cdot\varphi_{-\frac{1}{2}}(Y_0) = \varphi_0^4(Y_0),\ \&c.;\end{aligned}
\end{array}\right.$$

φ^n representing the result of the n successive functional operations indicated in each third member.

From the well-known development

$$(6)\quad y_h = y_0 + h\Delta_{\pm\frac{1}{2}} + \frac{h(h\mp 1)}{2}\Delta_0^2 + \frac{(h+1)h(h-1)}{2\,.\,3}\Delta^3_{\pm\frac{1}{2}} + \frac{(h+1)h(h-1)(h\mp 2)}{2\,.\,3\,.\,4}\Delta_0^4 + \&c.,$$

is directly obtained

$$(7)\qquad \varphi_{\pm\frac{1}{2}} = \delta_{\pm\frac{1}{2}} = k\left[\Delta_{\pm\frac{1}{2}} - \frac{1-k^2}{4\,.\,2\,.\,3}\Delta^3_{\pm\frac{1}{2}} + \frac{1-k^2}{4\,.\,2\,.\,3}\cdot\frac{3^2-k^2}{4\,.\,4\,.\,5}\Delta^5_{\pm\frac{1}{2}} - \&c.\right],$$

which is a *linear function* of certain differences, which are themselves *linear functions;* therefore φ^n is obtained by n symbolical multiplications of the series (7) according to the law (3). The subscript numbers are seen to result so that, omitting them from the operation, we have $\delta^n =$ *the symbolical n^{th} power of* δ; that is

$$(8)\qquad \delta = k\left[\Delta - \frac{1-k^2}{24}\Delta^3 + \frac{1-k^2}{24}\cdot\frac{9-k^2}{80}\Delta^5 \ldots\right],$$

(9) $$\delta^2 = k^2 \left[\Delta^2 - \frac{1-k^2}{12}\Delta^4 + \frac{1-k^2}{12}\cdot\frac{4-k^2}{30}\Delta^6 \ldots\right],$$

(10) $$\delta^3 = k^3 \left[\Delta^3 - \frac{1-k^2}{8}\Delta^5 + \frac{1-k^2}{8}\cdot\frac{37-13k^2}{240}\Delta^7 \ldots\right],$$

(11) $$\delta^4 = k^4 \left[\Delta^4 - \frac{1-k^2}{6}\Delta^6 + \frac{1-k^2}{6}\cdot\frac{7-3k^2}{40}\Delta^8 \ldots\right],$$

(12) $$\delta^5 = k^5 \left[\Delta^5 - \frac{5}{24}(1-k^2)\Delta^7 \ldots\right].$$

It will be observed that the notation δ^{2n}_m was substituted in (4) for each difference of the form ϑ^{2n}_{mi}, because the symbolic even powers of δ as there defined are *always* identical with ϑ^{2n}_{mi}. When i is odd, δ^{2n+1} is also identical with the middle ϑ^{2n+1} between Y_0 and Y_1; and all the series (8) . . . (12) may be used directly. When i is even, let δ'^{2n+1} denote half the sum of the two ϑ^{2n+1}s next above and below the middle point between Y_0 and Y_1; then in each case it will be the same function of δ^{2n}_0 and *its* differences, that δ' is of Y_0 and its differences; that is, if

(13) $$\delta' = \varphi'(Y_0), \qquad \text{then} \qquad \delta'^{2n+1} = \varphi'(\delta^{2n}_0).$$

But we have for these functions of k, *whether $\frac{1}{2}$ i is odd or even*,

(14) $$\delta'(k) = \tfrac{1}{2}\delta(2k); \qquad \text{that is,} \qquad \varphi'(k) = \tfrac{1}{2}\varphi_{\frac{1}{2}}(2k);$$

therefore the series for δ'^{2n+1} may be obtained by substituting $2k$ for k in (7) and symbolically multiplying this into the series for $\frac{1}{2}\delta^{2n}$. This gives

(15) $$\delta' = k\left[\Delta - \frac{1-4k^2}{24}\Delta^3 + \frac{1-4k^2}{24}\cdot\frac{9-4k^2}{80}\Delta^5 \ldots\right],$$

(16) $$\delta'^3 = k^3\left[\Delta^3 - \frac{1-2k^2}{8}\Delta^5 + \left(\frac{1-2k^2}{8}\cdot\frac{19-12k^2}{120} - \frac{1}{1920}\right)\Delta^7 \ldots\right],$$

(17) $$\delta'^5 = k^5\left[\Delta^5 - \frac{5-8k^2}{24}\Delta^7 \ldots\right].$$

Besides the symmetrical relations obtained above by introducing the functions δ, the following may be obtained directly from (6):—

(18) $$\vartheta_{\frac{1}{2}} = \Theta_{\frac{1}{2}} - \Theta^2_0, \qquad \vartheta_{i-\frac{1}{2}} = \Theta_{\frac{1}{2}} + \Theta^2_1;$$

(19) $$\begin{cases} \vartheta_{\frac{1}{2}} + \vartheta_{-\frac{1}{2}} = \Theta_{\frac{1}{2}} + \Theta_{-\frac{1}{2}}, \\ \vartheta_{\frac{1}{2}} - \vartheta_{-\frac{1}{2}} = \delta^2_0; \end{cases}$$

in which

(20) $$\Theta_{\pm\frac{1}{2}} = k\left[\Delta_{\pm\frac{1}{2}} - \frac{1-k^2}{2\,.\,3}\Delta^3_{\pm\frac{1}{2}} + \frac{1-k^2}{2\,.\,3}\cdot\frac{2^2-k^2}{4\,.\,5}\Delta^5_{\pm\frac{1}{2}} - \text{\&c.}\right],$$

$$(21) \qquad \Theta_n^2 = \tfrac{1}{2} k (1-k) \left[\Delta_n^2 - \frac{1+k}{3} \cdot \frac{2-k}{4} \Delta_n^4 + \frac{1+k}{3} \cdot \frac{2-k}{4} \cdot \frac{2+k}{5} \cdot \frac{3-k}{6} \Delta_n^6 - \&c. \right].$$

Since the functions involved in (19) need not be restricted to integral values of i, similar expressions may be obtained for the differences between Y_0 and $y_{\pm m}$ by substituting $m\,k$ for k in (9) and (20). By arranging these according to powers of $m\,k$ and using their differences taken with respect to m, algebraic expressions may easily be got for any of the differences ϑ_h^n. To do this let m^r be substituted for y_m in the series (4); and, $\nabla^n m^r$ being one of the n^{th} order of differences of the new series, let $\vartheta_h^n\,(m)$ be the *correspondingly placed* difference of the series y_m: then

$$(22) \qquad \begin{cases} \vartheta_h^n(-m) = \Phi_0^{[n]}(m) - \Psi_0^{[n]}(m), \\ \vartheta_h^n(m) \quad\; = \Phi_0^{[n]}(m) + \Psi_0^{[n]}(m); \end{cases}$$

in which, putting

$$(23) \qquad \Delta_0^{2n-1} = \tfrac{1}{2}(\Delta_{-\frac{1}{2}}^{2n-1} + \Delta_{\frac{1}{2}}^{2n-1}) = \Delta_{-\frac{1}{2}}^{2n-1} + \tfrac{1}{2}\Delta_0^{2n},$$

$$(24) \qquad \begin{cases} \Phi_0^{[n]}(m) = \nabla^n m \,.\, k(\Delta_0 - \frac{1}{6}\Delta_0^3 + \frac{1}{30}\Delta_0^5) + \nabla^n m^3 \,.\, \dfrac{k^3}{6}(\Delta_0^3 - \frac{1}{4}\Delta_0^5) + \nabla^n m^5 \,.\, \dfrac{k^5}{120}\Delta_0^5 + \&c., \\ \qquad\qquad = \nabla^n m \,.\, k\Delta_0 - \left(\dfrac{\nabla^n m}{k^2} - \nabla^n m^3\right) \cdot \dfrac{k^3}{6}(\Delta_0^3 - \frac{1}{5}\Delta_0^5) - \left(\dfrac{\nabla^n m^3}{k^2} - \nabla^n m^5\right) \cdot \dfrac{k^5}{120}\Delta_0^5 + \&c., \\ \Psi_0^{[n]}(m) = \nabla^n m^2 \,.\, \dfrac{k^2}{2}(\Delta_0^2 - \frac{1}{12}\Delta_0^4 + \frac{1}{90}\Delta_0^6) + \nabla^n m^4 \,.\, \dfrac{k^4}{24}(\Delta_0^4 - \frac{1}{6}\Delta_0^6) + \nabla^n m^6 \,.\, \dfrac{k^6}{720}\Delta_0^6 + \&c. \end{cases}$$

Formula (22) is a *perfectly general connection between the original and the interpolated differences of any order.*

When $n = 1$, (22) gives the means of deriving all the first differences,* ϑ_h, by computing the functions (24) for $\frac{i}{2}$ or $\frac{i-1}{2}$ different values of m, according as i is even or odd, — in the latter case obtaining $\vartheta_{\frac{i}{2}} = \delta_{\frac{1}{2}}$ from (8). For this method the second form of Φ is adapted. But methods of this sort, however simplified in numerical application, seem unnecessarily laborious.

Any interpolation may be made by combining (8) or (15) with (9), as follows. Compute the δ^2 opposite each value of Y, and the δ or δ' opposite each interval. Then from the former fill in all the second differences, ϑ_h^2, and by their means the first differences, ϑ_h, and thence the function itself. If i is even, the two differences $\vartheta_{\frac{i-1}{2}}$ and $\vartheta_{\frac{i+1}{2}}$, of which δ' is half the sum, must be derived from δ' and the second difference, $\vartheta_{\frac{i}{2}}^2$, opposite it, by taking the sum and difference $\delta' \mp \frac{1}{2}\vartheta_{\frac{i}{2}}^2$. This method is sometimes the most convenient, especially when $i = 5$. If the higher differences are

* For another method of this sort, with a different system of differences, see Mr. Ferrel's article on *Interpolation* in the *Mathematical Monthly*, Vol. III. p. 377.

very large, the ϑ^2_hs must themselves be interpolated, either by a distinct operation, or by the use of (10) or (16) and (11).*

An objection to these processes, when many interpolated values are to be inserted, is that the sum of the computed first differences will not generally be exactly equal to the differences between two successive values of Y; nor will the sum of the computed second differences exactly fill in the computed first differences, etc. These difficulties have to be remedied by subsequent arbitrary correction, or by computing the higher differences to one or more decimal places beyond those of Y. For this reason, interpolation to fourths, sixths, ninths, &c. should generally be done by successive interpolations to halves and thirds; and the choice of the special method of doing this must depend more or less upon the particular character of the function to be interpolated. The following are some practical methods with the numerical coefficients in the various series required.

Interpolation to Halves.

(21″) $$\Theta^2_n = \tfrac{1}{8}\left[\Delta^2_n - \tfrac{3}{16}\Delta^4_n + \tfrac{3}{16}\cdot\tfrac{5}{24}\Delta^6_n \ldots\right],$$

(15″) $$\delta' = \tfrac{1}{2}\Delta,$$

(9″) $$\delta^2 = \tfrac{1}{4}\left[\Delta^2 - \tfrac{1}{16}\Delta^4 + \tfrac{1}{16}\cdot\tfrac{1}{8}\Delta^6 \ldots\right].$$

The interpolated differences may be obtained in successive pairs by taking the half-sum and half-difference of the quantities

(25) $$\begin{cases} \vartheta_{\frac{3}{2}} + \vartheta_{\frac{1}{2}} = \Delta = 2\,\delta', \\ \vartheta_{\frac{3}{2}} - \vartheta_{\frac{1}{2}} = \Theta^2_0 + \Theta^2_1 = \vartheta^2_1. \end{cases}$$

When ϑ^2_1 can readily be obtained by interpolating the δ^2_hs, it may be more convenient to use (9″) instead of (21″) to obtain it.

Interpolation to Thirds.

(20‴) $$\Theta_{\pm\frac{1}{2}} = \tfrac{1}{3}\left[\Delta_{\pm\frac{1}{2}} - \tfrac{4}{27}\Delta^3_{\pm\frac{1}{2}} + \tfrac{4}{27}\cdot\tfrac{7}{36}\Delta^5_{\pm\frac{1}{2}} \ldots\right],$$

(21‴) $$\Theta^2_n = \tfrac{1}{9}\left[\Delta^2_n - \tfrac{5}{27}\Delta^4_n + \tfrac{5}{27}\cdot\tfrac{28}{135}\Delta^6_n \ldots\right];$$

(8‴) $$\delta = \tfrac{1}{3}\left[\Delta - \tfrac{1}{27}\Delta^3 + \tfrac{1}{27}\cdot\tfrac{1}{3}\Delta^5 \ldots\right],$$

(9‴) $$\delta^2 = \tfrac{1}{9}\left[\Delta^2 - \tfrac{2}{27}\Delta^4 + \tfrac{2}{27}\cdot\tfrac{7}{54}\Delta^6 \ldots\right].$$

* This general method, *as far as relates to odd values of i*, with algebraic forms for $\delta^{2n-1}_{\frac{1}{2}}$ and δ^{2n}_0 up to $n = 4$, is given by Encke in the *Astronomische Nachrichten*, Band 29, No. 695. Also a partial statement, obtained by *induction* apparently, of the symbolic connection of the series (8) . . . (12).

Formula (19) is sufficient to do the whole interpolation, using (20‴) and (9‴). But it is often better (especially when, in interpolating to sixths or twelfths, it is desirable not to write down the inserted values of the function themselves, till the whole is finished, but only their differences) to use (8‴) and (21‴), as follows:—

(26) $$\begin{cases} \vartheta_{\frac{5}{2}} + \vartheta_{\frac{1}{2}} = \Delta - \delta, & \qquad \vartheta_{\frac{3}{2}} = \delta, \\ \vartheta_{\frac{5}{2}} - \vartheta_{\frac{1}{2}} = \Theta_0^2 + \Theta_1^2. \end{cases}$$

Interpolation to Fourths.

(20^IV) $$\Theta_{\pm\frac{1}{2}} = \tfrac{1}{4}\left[\Delta_{\pm\frac{1}{2}} - \tfrac{5}{32}\Delta^3_{\pm\frac{1}{2}} + \tfrac{5}{32}\cdot\tfrac{63}{320}\Delta^5_{\pm\frac{1}{2}}\ \ldots\right],$$

(15^IV) $$\delta' = \tfrac{1}{4}\left[\Delta - \tfrac{1}{32}\Delta^3 + \tfrac{1}{32}\cdot\tfrac{7}{64}\Delta^5 \ldots\right],$$

(9^IV) $$\delta^2 = \tfrac{1}{16}\left[\Delta^2 - \tfrac{5}{64}\Delta^4 + \tfrac{5}{64}\cdot\tfrac{21}{160}\Delta^6 \ldots\right],$$

(11^IV) $$\delta^4 = \tfrac{1}{256}\left[\Delta^4 - \tfrac{5}{32}\Delta^6 \ldots\right].$$

Instead of interpolating twice to halves, $\vartheta_{\frac{1}{2}}$ and $\vartheta_{\frac{7}{2}}$ may be got by (19), using (20^IV) and (9^IV), and $\vartheta_{\frac{3}{2}}$ and $\vartheta_{\frac{5}{2}}$ by

(27) $$\begin{cases} \vartheta_{\frac{5}{2}} + \vartheta_{\frac{3}{2}} = 2\,\delta' = \Delta - (\vartheta_{\frac{1}{2}} + \vartheta_{\frac{7}{2}}), \\ \vartheta_{\frac{5}{2}} - \vartheta_{\frac{3}{2}} = \vartheta_2^2. \end{cases}$$

Either the second or third member of the first equation may be used; and ϑ_2^2 may be got by interpolating δ^2 to halves, using for this purpose (11^IV) if necessary.

Interpolation to Fifths.

(20^V) $$\Theta_{\pm\frac{1}{2}} = \tfrac{1}{5}\left[\Delta_{\pm\frac{1}{2}} - \tfrac{4}{25}\Delta^3_{\pm\frac{1}{2}} + \tfrac{4}{25}\cdot\tfrac{99}{500}\Delta^5_{\pm\frac{1}{2}}\ \ldots\right],$$

(8^V) $$\delta = \tfrac{1}{5}\left[\Delta - \tfrac{1}{25}\Delta^3 + \tfrac{1}{25}\cdot\tfrac{14}{125}\Delta^5 \ldots\right],$$

(9^V) $$\delta^2 = \tfrac{1}{25}\left[\Delta^2 - \tfrac{2}{25}\Delta^4 + \tfrac{2}{25}\cdot\tfrac{33}{250}\Delta^6 \ldots\right],$$

(10^V) $$\delta^3 = \tfrac{1}{125}\left[\Delta^3 - \tfrac{3}{25}\Delta^5 + \tfrac{3}{25}\cdot\tfrac{19}{125}\Delta^7 \ldots\right],$$

(11^V) $$\delta^4 = \tfrac{1}{625}\left[\Delta^4 - \tfrac{4}{25}\Delta^6 \ldots\right].$$

If, besides getting $\vartheta_{\frac{5}{2}} = \delta$ from (8^V), we use (20^V) and (9^V) in (19) to get $\vartheta_{\frac{1}{2}}$ and $\vartheta_{\frac{9}{2}}$, there will remain only $\vartheta_{\frac{3}{2}}$ and $\vartheta_{\frac{7}{2}}$ to be filled in.

Interpolation to Sixths.

$$(20^{\mathrm{VI}}) \qquad \Theta_{\pm\frac{1}{2}} = \tfrac{1}{6}\left[\Delta_{\pm\frac{1}{2}} - \tfrac{35}{216}\Delta^3_{\pm\frac{1}{2}} + \tfrac{35}{216}\cdot\tfrac{143}{720}\Delta^5_{\pm\frac{1}{2}}\ldots\right],$$

$$(15^{\mathrm{VI}}) \qquad \delta' = \tfrac{1}{6}\left[\Delta - \tfrac{1}{27}\Delta^3 + \tfrac{1}{27}\cdot\tfrac{1}{9}\Delta^5\ldots\right],$$

$$(9^{\mathrm{VI}}) \qquad \delta^2 = \tfrac{1}{36}\left[\Delta^2 - \tfrac{35}{432}\Delta^4 + \tfrac{35}{432}\cdot\tfrac{143}{1080}\Delta^6\ldots\right],$$

$$(16^{\mathrm{VI}}) \qquad \delta'^3 = \tfrac{1}{216}\left[\Delta^3 - \tfrac{17}{144}\Delta^5 + \left(\tfrac{17}{144}\cdot\tfrac{7}{45} - \tfrac{1}{1920}\right)\Delta^7\ldots\right],$$

$$(11^{\mathrm{VI}}) \qquad \delta^4 = \tfrac{1}{1296}\left[\Delta^4 - \tfrac{35}{216}\Delta^6\ldots\right].$$

If $\vartheta_{\frac{1}{2}}$ and $\vartheta_{\frac{11}{2}}$ are computed by (19), and $\vartheta_{\frac{5}{2}}$ and $\vartheta_{\frac{7}{2}}$ by using (15^{VI}) and (9^{VI}), there will remain only $\vartheta_{\frac{3}{2}}$ and $\vartheta_{\frac{9}{2}}$ to be filled in.

It will be observed that the series (8) . . . (12) and (15) . . . (17) always converge more rapidly than (20) and (21); and their use should sometimes be preferred on this account. A little reflection will often introduce, instead of the above-written fractions, some simpler equivalents, which may be mentally applied to the requisite degree of accuracy. Thus, in the examples given below, the following have occurred:—

$$\tfrac{3}{16} = \tfrac{1}{5} - \tfrac{1}{80}, \qquad \tfrac{7}{36} = \tfrac{1}{5} - \tfrac{1}{180},$$

$$\tfrac{4}{27} = \tfrac{1}{7} + \tfrac{1}{189}, \qquad \tfrac{14}{125} = \tfrac{1}{9} + \tfrac{1}{1125};$$

in all of which the right-hand term is easily applied when not small enough to be neglected.

The function Y, in the examples given below, is the Moon's declination for 1865, omitting the degrees; the values are taken where their higher differences are quite large. Two methods of interpolating to sixths (by halves and thirds, and then by thirds and halves) are given, and their results may be compared. For computing a lunar ephemeris, the first method here given has been found to possess peculiar advantages. An example of interpolating to fifths, i. e. to every tenth of a day, is then given on Encke's plan. The whole of the work which it is necessary to write down is printed. The columns headed S . 8 Θ^2, &c. contain the sums of each consecutive pair of values of the functions 8 Θ^2, &c. Since the sum and difference of any two numbers must be both odd or both even, in using (25) and (26) the nearest odd or even value has been taken for S . Θ^2, according as Δ and $\Delta - \delta$ were odd or even; and similarly with δ^2 and ϑ_1^2 in using (19) and (9″). In the interpolation to fifths, the computed third and second differences, ϑ^3 and ϑ^2, happen *exactly* to fill in the second and first respectively, all being carried to hundredths of a second, i. e. one place farther than the function Y.

Date.	Funct., Y.	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Functions for Interpolation to Halves by (25).			
							$\frac{3}{16}\Delta^4$	$8\,\Theta^2$	$S.8\,\Theta^2$	$S.\Theta^2$
d	′ ″	′ ″	′ ″	″	″	″	″	″	″	″
Apr. 27.5	38 2.6	+20 54.2	—18 7.9	+23.7	+27.9	—4.1	+5.2	—1093.1	—2161.8	—270.2
28.0	58 56.8	+ 3 10.0	17 44.2	47.5	23.8	5.0	4.5	1068.7	2088.9	261.2
28.5	62 6.8	—13 46.7	16 56.7	66.3	18.8	6.1	3.5	1020.2	1973.0	246.7
29.0	48 20.1	—29 37.1	15 50.4	+79.0	12.7	—5.3	2.4	952.8	—1825.6	—228.1
29.5	18 43.0		—14 31.4		+ 7.4		+1.4	— 872.8		

Functions for Interpolation to Thirds by (19).							
Date.	$\Delta^3-\frac{7}{36}\Delta^5$	$3\,\Theta$	$S.3\,\Theta$	$\frac{2}{27}\Delta^4$	$9\,\delta^2$	$S.\Theta$	δ^2
d	″	′ ″	′ ″	″	″	′ ″	″
Apr. 27.5	+24.5	+20 50.6					
28.0	48.5	+ 3 2.8	+23 53.4	+1.8	—1066.0	+ 7 57.8	—118.4
28.5	67.5	—13 56.7	—10 53.9	1.4	1018.1	— 3 38.0	113.2
29.0	+80.0	—29 49.0	—43 45.7	+0.9	— 951.3	—14 35.2	—105.8
29.5							

Functions for Interpolation to Fifths by (8^{v}), (9^{v}), and (10^{v}).					
Date.	$\Delta^3-\frac{14}{125}\Delta^5$	$5\,\delta$	$\frac{2}{25}\Delta^4$	$25\,\delta^2$	$125\,\delta^3$
d	″	′ ″	″	″	″
Apr. 27.5	+24.2	+20 53.23	+2.2	—1090.1	+24.2
28.0	48.1	+ 3 8.08	1.9	1066.1	48.1
28.5	67.0	—13 49.38	1.5	1018.2	67.0
29.0	+79.6	—29 40.28	1.0	951.4	+79.6
29.5			+0.6	— 872.0	

After Interpolation to Halves by (25).					For Interpolation to every 2 Hours by (26).			
Date.	Δ	Δ^2	Δ^3	Δ^4	δ	$S.9\,\Theta^2$	$\Delta-\delta$	$S.\Theta^2$
d h	′ ″	″	″	″	′ ″	″	′ ″	″
Apr. 27 12	+12 42.2							
27 18	8 12.0	—270.2	+3.8	+1.6	+2 44.0	—537.2	+5 28.0	—59.6
28 0	+ 3 45.6	266.4	5.2	1.4	+1 15.1	528.2	+2 30.5	58.7
28 6	— 0 35.6	261.2	6.8	1.6	—0 12.0	516.1	—0 23.6	57.4
28 12	4 50.0	254.4	7.7	0.9	1 36.8	501.5	3 13.2	55.8
28 18	8 56.7	246.7	8.9	1.2	2 59.0	484.9	5 57.7	53.9
29 0	12 54.5	237.8	+9.7	0.8	—4 18.3	—466.1	—8 36.2	—51.8
29 6	—16 42.6	—228.1		+0.4				
29 12								

After Interpolation to Thirds by (19).						For Interpolation to every 2 Hours by (9″).	
Date.	ϑ	Function.	Δ	$Δ^2$	$Δ^3$	$δ^2$	$ϑ_1^2$
d h	′ ″	′ ″	′ ″	″	″	″	″
Apr. 27 20		53 58.7					
	+4 58.1		+4 58.1				
28 0		58 56.8		—118.4		—29.6	
	+2 59.7		2 59.7		+1.4		—29.5
4		61 56.5		117.0		29.3	
			+1 2.7		1.9		29.1
8		62 59.2		115.1		28.8	
	—0 52.4		—0 52.4		1.9		28.6
12		62 6.8		113.2		28.3	
	2 45.6		2 45.6		2.4		28.0
16		59 21.2		110.8		27.7	
			4 36.4		2.5		27.4
20		54 44.8		108.3		27.1	
	6 24.7		6 24.7		+2.5		—26.7
29 0		48 20.1		—105.8		—26.4	
	—8 10.5		—8 10.5				
29 4		40 9.6					

	Interpolation by using (25), and then (26).		Interpolation by using (19), and then (9″).	
Date.	ϑ	Function.	ϑ	Function.
d h	′ ″	′ ″	′ ″	′ ″
Apr. 28 0		58 56.8		58 56.8
	+1 44.6		+1 44.6	
2		60 41.4		60 41.4
	1 15.1		1 15.1	
4		61 56.5		61 56.5
	0 45.9		0 45.9	
6		62 42.4		62 42.4
	+0 16.9		+0 16.8	
8		62 59.3		62 59.2
	—0 12.0		—0 11.9	
10		62 47.3		62 47.3
	0 40.5		0 40.5	
12		62 6.8		62 6.8
	1 8.7		1 8.8	
14		60 58.1		60 58.0
	1 36.8		1 36.8	
16		59 21.3		59 21.2
	2 4.5		2 4.5	
18		57 16.8		57 16.7
	2 31.9		2 31.9	
20		54 44.9		54 44.8
	2 59.0		2 59.0	
22		51 45.9		51 45.8
	—3 25.8		—3 25.7	
29 0		48 20.1		48 20.1

Interpolation to Fifths by (10^{v}), (9^{v}), and (8^{v}).				
Date.	$ϑ^3$	$ϑ^2$	ϑ	Function.
d	″	″	′ ″	′ ″
Apr. 27.9		—42.91		56 12.2
	+0.27		+2 44.57	
28.0		42.64*		58 56.8
	0.31		2 1.93	
28.1		42.33		60 58.7
	0.35		1 19.60	
28.2		41.98		62 18.3
	0.38*		+0 37.62*	
28.3		41.60		62 55.9
	0.42		—0 3.98	
28.4		41.18		62 52.0
	0.45		0 45.16	
28.5		40.73*		62 6.8
	0.48		1 25.89	
28.6		40.25		60 40.9
	0.51		2 6.14	
28.7		39.74		58 34.8
	0.54*		2 45.88*	
28.8		39.20		55 48.9
	0.56		3 25.08	
28.9		38.64		52 23.8
	0.58		4 3.72	
29.0		38.06*		48 20.1
	+0.60		—4 41.78	
29.1		—37.46		43 38.3

* These are the independent values derived from the original differences.